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(Residential Autonomous College under University of Calcutta)

M.A./M.SC. FOURTH SEMESTER EXAMINATION, MAY 2012

SECOND YEAR

Date : 25/05/2012 MATHEMATICS

Time : 11 am – 2 pm Paper : XIV Full Marks : 60

[Use separate Answer Books for each group]

<u>Group – A</u>

Answer **<u>Question No. 1</u>** and **<u>any three</u>** from the rest (symbols have their usual meanings):

1. Answer **<u>any one</u>** :
   a) Define Canonical transformation. Prove that a transformation is Canonical if there exists a gnerating function F such that $\frac{dF}{dt} = L - \bar{L}$, where L and $\bar{L}$ are Lagrangians corresponding to old and new coordinates respectively. [6]
   b) Derive Hamilton-Jacobi equation in the form $H\left(q_1, q_2 \ldots . q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots . \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0$. What is the physical significance of S? [6]

2. If the transformation equation between two sets of coordinates are $P = 2(1+q^{1/2}\cos p)q^{1/2}\sin p$ and $Q = \log(1+q^{1/2}\cos p)$, then show that the transformation is Canonical and the generating function is $F_3(p,Q) = -(e^Q - 1)^2 \tan p$. [8]

3. Prove Jacobi-Poisson Theorem that if u and v are both constants of motion, then their Poisson Bracket [u, v] is also a constant of motion. [8]

4. A rigid body is free to rotate about its centre of gravity O, the principal moments of inertia at which are 7, 25 and 32 units respectively. The body is given an angular velocity Ω about a line through the centre of gravity whose direction ratios are 4 : 0 : 3.
   Show that after time t, the components of angular velocity about the principal axes of inertia at O are $\frac{4}{5}\Omega\cos\phi$, $\frac{4}{5}\Omega\sin\phi$, $\frac{3}{5}\Omega\cos\phi$ where $\tan\frac{\phi}{2} = \tan h\left(\frac{3\Omega t}{10}\right)$.
   Deduce that ultimately the body rotates about the principal axis of intermediate moment of inertia. [8]

5. Show that the potential at an internal point P at a distance r from the centre of a solid sphere of radius 'a' and density ρ is
   $$\gamma 2\pi\rho\left(a^2 - \frac{r^2}{3}\right).$$
   Use the above result to show that if a self attracting sphere of uniform density and radius 'a' changes to another sphere of radius 'b' (b < a) and uniform density, the work done by its mutual attractive forces is
   $\frac{3}{5}\gamma M^2\left(\frac{1}{b} - \frac{1}{a}\right)$ where M is the mass of the sphere. [8]

6. Find the distribution of matter inside and outside a sphere of radius 'a' and also on its surface, if for a point distant r from the centre, the potential is given by $v_1 = \lambda(2a^3 - r^3)$ $(r < a)$ and $v_2 = \lambda\frac{a^4}{r}$ $(r > a)$. [8]

Group - B

1. Answer **any three** : [3×4]

a) If H_n is a Hermite polynomial of degree n, prove that $\int_{-\infty}^{\infty} e^{-x^2} H_n^{\,2}(x)dx = 2^n \underline{|n}\sqrt{\pi}$.

b) Establish Kummer's theorem : $F(\alpha,\beta;\beta-\alpha+1;-1) = \dfrac{\Gamma(\beta-\alpha+1)\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma(\beta+1)\Gamma\left(\frac{\beta}{2}-\alpha+1\right)}$, for hyergeometric function.

c) Show that $\lim\limits_{a\to\infty} {}_2F_1\left(1,a;1;\frac{x}{a}\right) = e^x$.

d) Prove that $\int_0^{\infty} e^{-x} L_n(x)L_m(x)dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$, where $L_K(x)$ is a Laguerre polynomial of degree K.

e) Expand x^3+x^2-3x+2 in a series of Laguerre polynomials.

Group C

2. Answer **any three** : [3×6]

a) If P(x, y, z) be any point in the bounded region R and S be the surface of a sphere of radius r having centre at P lying entirely within R, then prove that $u(P) = \dfrac{1}{4\pi r^2}\iint\limits_S u(Q)ds$, where u is harmonic in R and Q be a point on S.

b) Solve the partial differential equation $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 u}{\partial\varphi^2} = 0$ by the method of separation of variables.

c) Solve the heat equation $\dfrac{\partial T}{\partial t} = \alpha\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right)$, $0 \le x \le a$, $0 \le y \le b$, $t \ge 0$ with the boundary conditions $T(0,y,t) = T(a,y,t) = 0$, $0 \le y \le b$, $t \ge 0$; $T(x,0,t) = T(x,b,t) = 0$, $0 \le x \le a$, $t \ge 0$ and the initial condition $T(x,y,0) = f(x,y)$, $0 \le x \le a$, $0 \le y \le b$.

d) Prove that spherically symmetric solution of wave equation $\nabla^2 u = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ with respect to the origin is $u = \dfrac{e^{\pm iKr}}{r}.e^{\pm icKt}$. Hence deduce Helmholtz's First Theorem in the form

$$\frac{1}{4\pi}\iint\left[-v\frac{\partial}{\partial n}\frac{e^{iK|\underset{\sim}{r}-\underset{\sim}{r_0}|}}{|\underset{\sim}{r}-\underset{\sim}{r_0}|} + \frac{e^{iK|\underset{\sim}{r}-\underset{\sim}{r_0}|}}{|\underset{\sim}{r}-\underset{\sim}{r_0}|}\frac{\partial v}{\partial n}\right]ds = \begin{cases} v(\underset{\sim}{r_0}), & \underset{\sim}{r_0} \in V \\ 0, & \underset{\sim}{r_0} \notin V \end{cases}.$$

e) Obtain the solution of the exterior Dirichlet problem for a circle described as $\nabla^2 u = 0$, $r \ge a$, $0 \le \theta \le 2\pi$; with the boundary condition $u(a,\theta) = f(\theta)$ in the form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-a^2)f(\varphi)}{r^2 - 2ar\cos(\varphi-\theta) + a^2}d\varphi.$$